Coloring the square of the Cartesian product of two cycles

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Abstract

The square G^2 of a graph G is defined on the vertex set of G in such a way that distinct vertices with distance at most 2 in G are joined by an edge. We study the chromatic number of the square of the Cartesian product $C_m \square C_n$ of two cycles and show that the value of this parameter is at most 7 except when (m, n) is $(3, 3)$, in which case the value is 9, and when (m, n) is $(4, 4)$ or $(3, 5)$, in which case the value is 8. Moreover, we conjecture that whenever $G = C_m \square C_n$, the chromatic number of G^2 equals $\lceil mn/\alpha(G^2) \rceil$, where $\alpha(G^2)$ denotes the maximum size of an independent set in G^2.

Key words: Chromatic number, square, distance-2 coloring, Cartesian product of cycles.

1 Introduction

A *proper k-coloring* of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a mapping c from $V(G)$ to the set $\{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever

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uv is an edge in $E(G)$. The *chromatic number* $\chi(G)$ of G is the smallest k for which G admits a proper k-coloring.

Let G and H be graphs. The *Cartesian product* $G\Box H$ of G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. Let P_n and C_n denote respectively the path and the cycle on n vertices. We will denote by $G_{m,n}$ the *grid* $P_m\Box P_n$ with m rows and n columns and by $T_{m,n}$ the *toroidal grid* $C_m\Box C_n$ with m rows and n columns.

The *square* G^2 of a graph G is given by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or u and v have a common neighbor. In other words, any two vertices within distance at most 2 in G are linked by an edge in G^2. Let $\Delta(G)$ denote the maximum degree of G. The problem of determining the chromatic number of the square of particular graphs has attracted very much attention, with a particular focus on the square of planar graphs (see e.g. [2, 5, 6, 11, 12]), following Wegner [15], who conjectured that every planar graph G with maximum degree at least 8 satisfies $\chi(G^2) \leq \lfloor \frac{3}{2}\Delta(G)\rfloor + 1$. Havet *et al.* proved in [5] that the square of any such planar graph admits a proper coloring using $(\frac{3}{2} + o(1))\Delta(G)$ colors.

In [1], Chiang and Yan studied the chromatic number of the square of Cartesian products of paths and cycles and proved the following:

Theorem 1 (Chiang and Yan [1]) *If $G = C_m\Box P_n$ with $m \geq 3$ and $n \geq 2$, then*

$$\chi(G^2) = \begin{cases} 4 & \text{if } n = 2 \text{ and } m \equiv 0 \pmod 4, \\ 6 & \text{if } n = 2 \text{ and } m \in \{3, 6\}, \\ 6 & \text{if } n \geq 3 \text{ and } m \not\equiv 0 \pmod 5, \\ 5 & \text{otherwise.} \end{cases}$$

Since $C_m\Box P_n$ is a subgraph of $C_m\Box C_n$, Theorem 1 provides lower bounds for the chromatic number of the square of toroidal grids.

A proper coloring of the square G^2 of a graph G is often called a *distance-2 coloring* of G. In [13], Pór and Wood studied the notion of $\mathcal{F}$-free coloring. Let $\mathcal{F}$ be a family of connected bipartite graphs, each with at least three vertices. An *$\mathcal{F}$-free coloring* of a graph G is then a proper vertex coloring of G with no bichromatic subgraph in $\mathcal{F}$. This notion generalizes several types of colorings and, in particular, distance-2 coloring when $\mathcal{F} = \{P_3\}$. They obtained an upper bound on the $\mathcal{F}$-free chromatic number of cartesian

products of general graphs. Moreover, in case of distance-2 coloring, they proved that the chromatic number of the square of any graph given as the Cartesian product of d cycles is at most $6d + O(\log d)$.

An $L(p, q)$-*labeling* of a graph G is an assignment ϕ of nonnegative integers to the vertices of G so that $|\phi(u) - \phi(v)| \geq p$ whenever u and v are adjacent and $|\phi(u) - \phi(v)| \geq q$ whenever u and v are at distance 2 in G. The λ_q^p-*number* of G is defined as the smallest k such that G admits an $L(p, q)$-labeling on the set $\{0, 1, \ldots, k\}$ (note that such a labeling uses $k + 1$ labels). It follows from the definition that any $L(1, 0)$-labeling of G is an ordinary proper coloring of G and that any $L(1, 1)$-labeling of G is a proper coloring of the square of G. Therefore, $\chi(G) = \lambda_0^1(G) + 1$ and $\chi(G^2) = \lambda_1^1(G) + 1$ for every graph G.

The notion of $L(p, q)$-labeling was introduced by Griggs and Yeh [4] to model the *Channel Assignment Problem*. They conjectured that $\lambda_1^2(G) \leq \Delta(G)^2$ for every graph G. This motivated many authors to study $L(2, 1)$-labeling of some particular classes of graphs, and the case of Cartesian products of graphs was investigated in [1, 3, 7, 8, 9, 10, 14, 16].

In particular, Schwartz and Troxell [14] considered $L(2, 1)$-labelings of products of cycles and proved the following:

Theorem 2 (Schwartz and Troxell [14]) *If $T_{m,n} = C_m \Box C_n$ with $3 \leq m \leq n$, then*

$$\lambda_1^2(T_{m,n}) = \begin{cases} 6 & \text{if } m, n \equiv 0 \pmod 7, \\ 8 & \text{if } (m, n) \in A, \\ 7 & \text{otherwise.} \end{cases}$$

where $A = \big\{(3, i) : i \in \{4, 10\} \text{ or } i = 2j + 1 \text{ with } j \in \mathbb{N}\big\} \cup \big\{(5, i) : i \in \{5, 6, 9, 10, 13, 17\}\big\} \cup \big\{(6, 7), (6, 11), (7, 9), (9, 10)\big\}$.

Since every $L(2, 1)$-labeling is an $L(1, 1)$-labeling, $\lambda_1^2(G) + 1 \geq \lambda_1^1(G) + 1 = \chi(G^2)$ for every graph G. Therefore, Theorem 2 provides upper bounds on the chromatic number of the square of toroidal grids (the upper bounds corresponding to the three cases of Theorem 2 are 7, 9, and 8, respectively).

Our main result will improve the bounds provided by Theorems 1 and 2 and by the general result of Pór and Wood [13]:

Theorem 3 *If $T_{m,n} = C_m \Box C_n$ with $3 \leq m \leq n$, then $\chi(T_{m,n}^2) \leq 7$ except $\chi(T_{3,3}^2) = 9$ and $\chi(T_{3,5}^2) = \chi(T_{4,4}^2) = 8$.*

2 Coloring the squares of toroidal grids

In this section, we shall prove Theorem 3 and give more precise bounds for Cartesian products of some particular cycles.

We shall construct explicit colorings using combinations of *patterns* given in matrix form. Each pattern can be thought of as a proper coloring of the square of the toroidal grid of the same size. For instance, the pattern E depicted in Figure 1 provides in an obvious way a proper 7-coloring of the square of $T_{3,7}$. Moreover, by repeating this pattern, we can easily obtain a proper 7-coloring of the square of toroidal grids of the form $T_{3m,7q}$.

Let G be a graph and c be a proper coloring of G. Since every color class under c is an independent set, we have the following standard observation:

Observation 4 $\chi(G) \geq \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil$ *where $\alpha(G)$ denotes the maximum size of an independent set in G.*

We shall extensively use a result of Sylvester. Given two integers r and s, let $S(r,s)$ denote the set of all nonnegative integer combinations of r and s:

$$S(r,s) = \{\alpha r + \beta s : \alpha, \beta \text{ nonnegative integers}\}.$$

Lemma 5 (Sylvester) *If r and s are relatively prime integers greater than 1, then $t \in S(r,s)$ for all $t \geq (r-1)(s-1)$, and $(r-1)(s-1)-1 \notin S(r,s)$.*

We then have:

Theorem 6 *If $T_{m,n} = C_m \Box C_n$ with $m \in S(4,7)$ and $n \in S(3,7)$, then $\chi(T^2_{m,n}) \leq 7$.*

Proof. Let $m \in S(4,7)$ and $n \in S(3,7)$. We use the following 7×7 pattern A to prove the claim.

$$A = \begin{array}{|ccccccc|}
\hline
1 & 6 & 4 & 2 & 7 & 5 & 3 \\
2 & 7 & 5 & 3 & 1 & 6 & 4 \\
3 & 1 & 6 & 4 & 2 & 7 & 5 \\
4 & 2 & 7 & 5 & 3 & 1 & 6 \\
5 & 3 & 1 & 6 & 4 & 2 & 7 \\
6 & 4 & 2 & 7 & 5 & 3 & 1 \\
7 & 5 & 3 & 1 & 6 & 4 & 2 \\
\hline
\end{array}$$

It is easy to check that this pattern properly colors $T^2_{7,7}$. For any pattern X, let X_i, X'_j be the subpatterns of X such that X_i is obtained by taking the i first rows of X and X'_j is obtained by taking the j first columns of X. It is again easy to check that the patterns A_4 and A'_3 provide proper colorings of $T^2_{4,7}$ and $T^2_{7,3}$, respectively. Therefore, using combinations of A and A_4, we can get a $m \times 7$ pattern Y. Moreover, using combinations of Y and Y'_3, we can get a $m \times n$ pattern that provides a proper 7-coloring of $T^2_{m,n}$, except when $(m,n) = (7a+4b, 7c+3d)$ with $a, c \geq 0$ and $b, d > 0$. In that case, it is enough to replace the color 4 in the upper-right corner of the rightmost copy of Y'_3 by 3 (see example below). ∎

For example, the following pattern B provides a proper 7-coloring of $T^2_{11,13}$, obtained from A by using the combinations $11 = 7+4$ and $13 = 7+2\times 3$ and replacing the color 4 by 3 in the upper-right corner.

$$B = \begin{array}{|ccccccc|ccc|ccc|}
\hline
1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 1 & 6 & \underline{\mathbf{3}} \\
2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 2 & 7 & 5 \\
3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 3 & 1 & 6 \\
4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 4 & 2 & 7 \\
5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 5 & 3 & 1 \\
6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 6 & 4 & 2 \\
7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 7 & 5 & 3 \\
\hline
1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 1 & 6 & 4 \\
2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 2 & 7 & 5 \\
3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 3 & 1 & 6 \\
4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 4 & 2 & 7 \\
\hline
\end{array}$$

By Lemma 5 we then get:

Corollary 7 *If $T_{m,n} = C_m \Box C_n$ with $m \geq 12$ and $n \geq 18$, then $\chi(T^2_{m,n}) \leq 7$.*

We now consider $T^2_{3,n}$.

Theorem 8 *If $T_{3,n} = C_3 \Box C_n$ with $n \geq 3$, then*

$$\chi(T^2_{3,n}) = \begin{cases} 6 & \text{if } n \text{ is even,} \\ 7 & \text{if } n \text{ is odd and } n \geq 7, \\ 8 & \text{if } n = 5, \\ 9 & \text{if } n = 3. \end{cases}$$

$$C = \begin{array}{|cccc|} \hline 1 & 4 & 2 & 5 \\ 2 & 5 & 3 & 6 \\ 3 & 6 & 1 & 4 \\ \hline \end{array} \qquad D = \begin{array}{|cccccc|} \hline 1 & 4 & 2 & 5 & 3 & 6 \\ 2 & 5 & 3 & 6 & 1 & 4 \\ 3 & 6 & 1 & 4 & 2 & 5 \\ \hline \end{array} \qquad E = \begin{array}{|ccccccc|} \hline 1 & 4 & 2 & 3 & 1 & 2 & 5 \\ 2 & 5 & 1 & 4 & 7 & 3 & 6 \\ 3 & 6 & 7 & 5 & 6 & 4 & 7 \\ \hline \end{array}$$

Figure 1: Patterns for Theorem 8

Proof. Let C, D, and E be the patterns given in Figure 1. These patterns clearly provide proper colorings of $T^2_{3,4}$, $T^2_{3,6}$, and $T^2_{3,7}$, respectively. For the upper bounds, we use the combinations of patterns C and D to obtain the even cases and use the combinations of patterns C, D, and E to obtain the odd cases. The remaining cases are $n \in \{3, 5, 9\}$, and the following patterns provide the required proper colorings of $T_{3,3}$, $T_{3,5}$, and $T_{3,9}$, respectively.

$$\begin{array}{|ccc|} \hline 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ \hline \end{array} \quad \begin{array}{|ccccc|} \hline 1 & 4 & 2 & 3 & 6 \\ 2 & 5 & 1 & 4 & 7 \\ 3 & 6 & 7 & 5 & 8 \\ \hline \end{array} \quad \begin{array}{|ccccccccc|} \hline 1 & 4 & 2 & 3 & 1 & 4 & 5 & 3 & 6 \\ 2 & 5 & 1 & 4 & 2 & 3 & 6 & 4 & 7 \\ 3 & 6 & 7 & 5 & 6 & 7 & 1 & 2 & 5 \\ \hline \end{array}$$

For the lower bounds, notice that the intersection of any independent set I in $T^2_{3,n}$ with any two consecutive columns contains at most one vertex. Therefore, $\alpha(T^2_{3,n}) \leq \lfloor n/2 \rfloor$. By Observation 4, $\chi(T^2_{3,n}) > 6$ when n is odd; also, $\chi(T^2_{3,n}) > 7$ when $n = 5$ and $\chi(T^2_{3,n}) \geq 9$ when $n = 3$. ∎

As in the proof of Theorem 6, we can obtain proper colorings of $T^2_{3k,n}$, for $k \geq 1$, by using combinations of the patterns given in Theorem 8. We thus get the following:

Corollary 9 *If $T_{3k,n} = C_{3k} \square C_n$ with $k \geq 1$ and $n \geq 3$, then*

$$\chi(T^2_{3k,n}) \leq \begin{cases} 6 & \text{if } n \text{ is even,} \\ 7 & \text{if } n \text{ is odd and } n \geq 7, \\ 8 & \text{if } n = 5, \\ 9 & \text{if } n = 3. \end{cases}$$

We now consider $T^2_{4,n}$.

$$F = \begin{array}{|ccc|} \hline 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 1 \\ 4 & 6 & 2 \\ \hline \end{array} \qquad G = \begin{array}{|ccccc|} \hline 1 & 3 & 2 & 4 & 6 \\ 2 & 4 & 6 & 3 & 5 \\ 3 & 5 & 7 & 2 & 1 \\ 4 & 6 & 1 & 5 & 7 \\ \hline \end{array}$$

$$H_1 = \begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 1 & 2 \\ \hline \end{array} \qquad H_2 = \begin{array}{|ccccccc|} \hline 1 & 3 & 2 & 6 & 4 & 7 & 5 \\ 2 & 4 & 5 & 7 & 1 & 3 & 6 \\ 3 & 1 & 6 & 2 & 5 & 4 & 7 \\ 4 & 5 & 7 & 1 & 3 & 6 & 2 \\ \hline \end{array}$$

Figure 2: Patterns for Theorem 10

Theorem 10 *If $T_{4,n} = C_4 \square C_n$ with $n \geq 3$, then*

$$\chi(T_{4,n}^2) = \begin{cases} 6 & \text{if } n \equiv 0 \pmod 3, \\ 8 & \text{if } n = 4, \\ 7 & \text{otherwise.} \end{cases}$$

Proof. For $m = 3k$, this follows from Corollary 9. Let F and G be the patterns given in Figure 2. These patterns clearly provide proper colorings of $T_{4,3}^2$ and $T_{4,5}^2$, respectively. Thanks to Lemma 5, by using combinations of F and G, we can get a proper 7-coloring of $T_{4,n}^2$ except when $n \in \{4, 7\}$. By using patterns H_1 and H_2 given in Figure 2, we obtain proper colorings of $T_{4,4}^2$ and $T_{4,7}^2$, respectively.

An independent set in $T_{4,n}^2$ has at most two vertices in any three consecutive columns. Thus, $\alpha(T_{4,n}^2) \leq \lfloor \frac{2n}{3} \rfloor$. By Observation 4, $\chi(T_{4,n}^2) > 6$ when n is not a multiple of 3 and $\chi(T_{4,n}^2) \geq 8$ when $n = 4$. ∎

Using combinations of the patterns from Theorem 10, we get the following:

Corollary 11 *If $T_{4k,n} = C_{4k} \square C_n$ with $k \geq 1$ and $n \geq 3$, then*

$$\chi(T_{4k,n}^2) \leq \begin{cases} 6 & \text{if } n \equiv 0 \pmod 3, \\ 8 & \text{if } n = 4, \\ 7 & \text{otherwise.} \end{cases}$$

$$I = \begin{array}{|ccccc|}\hline 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ \hline \end{array} \qquad J = \begin{array}{|cccccc|}\hline 6 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ \hline \end{array}$$

Figure 3: Patterns for Theorem 12

We now consider $T^2_{5,n}$.

Theorem 12 *If* $T_{5,n} = C_5 \square C_n$ *with* $n \geq 5$, *then*

$$\chi(T^2_{5,n}) = \begin{cases} 5 & \text{if } n \equiv 0 \pmod 5, \\ 7 & \text{if } n = 7, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Let I and J be the patterns given in Figure 3; they provide proper colorings of $T^2_{5,5}$ and $T^2_{5,6}$, respectively. We use combinations of I and J to get a proper 5-coloring (resp. 6-coloring) of $T^2_{5,n}$ when $n \equiv 0 \pmod 5$ (resp. when $n \in S(5,6)$).

The remaining cases are $n \in \{7, 8, 9, 13, 14, 16, 19\}$. The corresponding patterns are given below, except for $n = 16$, in which case the corresponding pattern is obtained by combining two 5×8 patterns.

$$\begin{array}{|ccccccc|}\hline 1 & 3 & 2 & 1 & 7 & 5 & 4 \\ 2 & 4 & 5 & 3 & 6 & 1 & 7 \\ 3 & 1 & 6 & 7 & 5 & 4 & 6 \\ 4 & 2 & 3 & 4 & 2 & 7 & 1 \\ 5 & 6 & 7 & 5 & 3 & 6 & 2 \\ \hline \end{array}$$
$n = 7$

$$\begin{array}{|cccccccc|}\hline 1 & 3 & 2 & 6 & 3 & 1 & 5 & 4 \\ 2 & 4 & 5 & 1 & 4 & 2 & 3 & 6 \\ 3 & 1 & 6 & 2 & 5 & 6 & 4 & 5 \\ 4 & 2 & 3 & 4 & 1 & 3 & 2 & 1 \\ 5 & 6 & 1 & 5 & 2 & 4 & 6 & 3 \\ \hline \end{array}$$
$n = 8$

$$\begin{array}{|ccccccccc|}\hline 1 & 3 & 5 & 1 & 4 & 6 & 5 & 2 & 4 \\ 2 & 4 & 6 & 3 & 2 & 1 & 4 & 3 & 5 \\ 3 & 1 & 2 & 4 & 6 & 5 & 2 & 1 & 6 \\ 4 & 6 & 3 & 5 & 1 & 4 & 3 & 5 & 2 \\ 5 & 2 & 4 & 6 & 3 & 2 & 1 & 6 & 3 \\ \hline \end{array}$$
$n = 9$

$$\begin{array}{|ccccccccccccc|}\hline 1 & 3 & 2 & 6 & 3 & 1 & 5 & 2 & 4 & 3 & 6 & 5 & 4 \\ 2 & 4 & 5 & 1 & 4 & 2 & 6 & 3 & 1 & 5 & 2 & 1 & 6 \\ 3 & 1 & 6 & 2 & 5 & 3 & 1 & 4 & 2 & 6 & 3 & 4 & 5 \\ 4 & 2 & 3 & 4 & 1 & 6 & 2 & 5 & 3 & 1 & 5 & 6 & 1 \\ 5 & 6 & 1 & 5 & 2 & 4 & 3 & 1 & 6 & 2 & 4 & 3 & 2 \\ \hline \end{array}$$
$n = 13$

1	3	2	6	3	1	5	2	4	3	2	5	6	4
2	4	5	1	4	2	6	3	5	6	4	1	3	5
3	1	6	2	5	3	1	4	2	1	5	2	4	6
4	2	3	4	1	6	2	5	3	4	6	3	5	1
5	6	1	5	2	4	3	1	6	5	1	4	2	3

$n = 14$

1	3	2	6	3	1	5	2	4	3	1	4	2	1	6	2	5	3	4
2	4	5	1	4	2	6	3	1	5	2	6	3	5	4	3	6	1	5
3	1	6	2	5	3	1	4	2	6	3	1	4	6	1	5	2	4	6
4	2	3	4	1	6	2	5	3	1	4	2	5	3	2	6	3	5	1
5	6	1	5	2	4	3	1	6	2	5	3	6	4	5	1	4	6	2

$n = 19$

An independent set in $T^2_{5,n}$ has at most one vertex in any column; thus $\alpha(T^2_{5,n}) \leq n$. Therefore, $\chi(T^2_{5,n}) \geq 5$ by Observation 4. It is easy to check that $\alpha(T^2_{5,n}) < n$ when n is not a multiple of 5 (and thus $\chi(T^2_{5,n}) > 5$) and that $\alpha(T^2_{5,7}) = 5$ (and thus $\chi(T^2_{5,7}) \geq \frac{35}{5} = 7$). ∎

Using combinations of the patterns from Theorem 12, we get the following:

Corollary 13 *If $T_{5k,n} = C_{5k} \Box C_n$ with $k \geq 1$ and $n \geq 5$, then*

$$\chi(T^2_{5k,n}) \leq \begin{cases} 5 & \text{if } n \equiv 0 \pmod 5, \\ 7 & \text{if } n = 7, \\ 6 & \text{otherwise.} \end{cases}$$

At this point, we are able to prove our main result.

Proof of Theorem 3. By Corollaries 9, 11, and 13, we already proved that if one of m, n is a multiple of 3, 4, or 5, then Theorem 3 holds. By Lemma 5 and Corollary 7, the remaining cases are 11×11, 13×13, 13×17 and 17×17. Let K be the 7×13 pattern given in Figure 4. As in the proof of Theorem 6, we use combinations of K and K_3 (the corresponding pattern, not the complete graph) to obtain an $m \times 13$ pattern X for $m \in S(7, 3)$. We then use combinations of X and X'_4 to obtain an $m \times n$ pattern for

$$K = \begin{array}{|ccccccccccccc|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 & 6 \\
3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \\
4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 \\
\hline
\end{array}$$

Figure 4: Pattern for Theorem 3

$n \in S(13, 4)$. In this way, we can obtain proper 7-colorings of $T^2_{13,13}$, $T^2_{17,13}$, and $T^2_{17,17}$. We simply transpose the 17×13 pattern to get a 13×17 pattern. Finally, an 11×11 pattern that properly 6-colors $T^2_{11,11}$ is as follows:

$$\begin{array}{|ccccccccccc|}
\hline
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 & 4 & 5 & 6 & 4 & 1 & 6 & 2 \\
5 & 1 & 2 & 3 & 1 & 2 & 3 & 5 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 & 6 & 4 & 1 & 6 & 4 & 5 & 1 \\
4 & 5 & 6 & 1 & 3 & 5 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 & 2 & 6 & 4 & 5 & 6 & 4 & 5 \\
6 & 4 & 1 & 6 & 5 & 3 & 1 & 2 & 3 & 1 & 2 \\
3 & 5 & 2 & 3 & 1 & 2 & 6 & 4 & 5 & 6 & 4 \\
1 & 6 & 4 & 5 & 6 & 4 & 5 & 1 & 2 & 3 & 5 \\
2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 1 & 6 \\
4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 & 2 & 3 \\
\hline
\end{array}$$

$\blacksquare$

As we have seen before, the general upper bound of 7 for $\chi(T^2_{m,n})$ given in Theorem 3 can be decreased for particular values of m and n. We now provide other cases for which this bound can be decreased to 6.

Using combinations of the 11×11 pattern above, we get:

Corollary 14 *If $T_{m,n} = C_m \square C_n$ with $m, n \geq 3$ and $m, n \equiv 0 \pmod{11}$, then $\chi(T^2_{m,n}) \leq 6$.*

The same bound can be obtained for $T^2_{6,n}$:

Theorem 15 *If $T_{6,n} = C_6 \square C_n$ with $n \geq 6$, then $\chi(T^2_{6,n}) = 6$.*

$$L = \begin{array}{|cccc|} \hline 1 & 3 & 6 & 4 \\ 2 & 4 & 1 & 5 \\ 3 & 5 & 2 & 6 \\ 4 & 6 & 3 & 1 \\ 5 & 1 & 4 & 2 \\ 6 & 2 & 5 & 3 \\ \hline \end{array} \qquad M = \begin{array}{|ccc|} \hline 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 1 \\ 4 & 6 & 2 \\ 5 & 1 & 3 \\ 6 & 2 & 4 \\ \hline \end{array}$$

Figure 5: Patterns for Theorem 15

Proof. Let L and M be the patterns given in Figure 5, which properly 6-color $T^2_{6,4}$ and $T^2_{6,2}$, respectively. By Lemma 5, we can get a proper 6-coloring of $T^2_{6,n}$ by using combinations of patterns L and M. ∎

Using combinations of the patterns from Theorem 15, we get the following:

Corollary 16 *If $T_{6k,n} = C_{6k} \square C_n$ with $k \geq 1$ and $n \geq 6$, then $\chi(T^2_{6k,n}) \leq 6$.*

Finally, using Corollary 13 and the lower bound given by Theorem 1, we get the following

Corollary 17 *If $T_{m,n} = C_m \square C_n$ with $m, n \geq 3$, then $\chi(T^2_{m,n}) \geq 5$. Moreover, $\chi(T^2_{m,n}) = 5$ if and only if $m, n \equiv 0 \pmod 5$.*

3 Discussion

In this paper, we investigated the chromatic number of the squares of toroidal grids; that is, squares of Cartesian products of two cycles. We obtained general upper bounds for this parameter by providing explicit proper colorings based on the use of specific patterns. This leads in an obvious way to a linear time algorithm for constructing such colorings.

Table 1 summarizes those of our results that give tight bounds. We also included two cases, marked by $(*)$, for which the tight bound has been obtained by a computer program. In all the cases for which a tight bound has been obtained, this bound matches the lower bound given by Observation 4. Therefore, we propose the following:

values of m and n	$\chi(T^2_{m,n})$
$m, n \equiv 0 \pmod 5$	5
$m = 3$, $n \equiv 0 \pmod 2$	6
$m = 4$, $n \equiv 0 \pmod 3$	6
$m = 6$, $n \geq 6$	6
$m = 8$, $n = 11, 13$ $(*)$	6
$m \equiv 0 \pmod 3$, $m \not\equiv 0 \pmod 5$, $n \equiv 0 \pmod 2$, $n \not\equiv 0 \pmod 5$	6
$m \equiv 0 \pmod 5$, $n \not\equiv 0 \pmod 5$, $n \geq 6$, $n \neq 7$	6
$m \equiv 0 \pmod 6$, $n \geq 6$, $n \not\equiv 0 \pmod 5$	6
$m, n \equiv 0 \pmod{11}$, $m \not\equiv 0 \pmod 5$, $n \not\equiv 0 \pmod 5$	6
$m = 3$, $n \not\equiv 0 \pmod 2$, $n \neq 3, 5$	7
$m = 4$, $n \not\equiv 0 \pmod 3$, $n \neq 4$	7
$m = 5$, $n = 7$	7
$m = 7$, $n = 7, 8$ $(*)$	7
$m = 3$, $n = 5$	8
$m = 4$, $n = 4$	8
$m = 3$, $n = 3$	9

Table 1: Summary of results on $\chi(T^2_{m,n})$

Conjecture 18 *For every toroidal grid $T_{m,n}$ with $m,n \geq 3$, $\chi(T^2_{m,n}) = \left\lceil \frac{|V(T^2_{m,n})|}{\alpha(T^2_{m,n})} \right\rceil$.*

Moreover, it is likely that the chromatic number of the squares of sufficiently large toroidal grids is at most 6. We therefore propose the following:

Conjecture 19 *The exists some constant c such that for every toroidal grid $T_{m,n}$ with $m,n \geq c$, $\chi(T^2_{m,n}) \leq 6$.*

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